JOURNAL OF APPROXIMATION THEORY 41, 91–99 (1984)

# Strong Uniqueness in Nonlinear Approximation

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*Communicated by E. W. Cheney*

Received September 18, 1981

## 1. INTRODUCTION

Let $X$ be a real or complex normed linear space and $K$ be an arbitrary subset of $X$. Recall that an element $\pi$ in $K$ is said to be a strongly unique element of best approximation (SUBA) of an element $f$ in $X$ if there exists a constant $r > 0$ such that for all $g$ in $K$,

$$\|f-g\| \geqslant \|f-\pi\| + r\,\|\pi - g\|.$$

For each $z \in X$ we define the set of supporting functionals at $z$ to be the set

$$\mathscr{L}_z = \{\phi \in X^*: \|\phi\| = 1 \text{ and } \phi(z) = \|z\|\}.$$

Consider the condition that an element $\pi$ in $K$ and $f$ in $X$ might satisfy: there exists a constant $r > 0$ such that for all $k$ in $K$

$$\sup_{\phi \in \mathscr{L}_{f-\pi}} \operatorname{Re} \phi(\pi - k) \geqslant r\,\|\pi - k\|, \tag{1.1}$$

where Re $\phi(g)$ denotes the real part of the functional $\phi(g)$. It was first shown in [7] (in a slightly different form than stated here) that if $K$ is a subspace in a real normed linear space then condition (1.1) is a necessary and sufficient condition for $\pi$ to be a SUBA to $f$ from $K$. This was subsequently extended to the case when $K$ is a subspace in a complex normed linear space in [1] and when $K$ is a convex set in [5]. We observe that condition (1.1) is always a sufficient condition for $\pi$ to be a SUBA to $f$, regardless of whatever property the set $K$ may or may not have. And so the interesting problem is to charaterize those sets $K$ for which (1.1) is also a necessary condition for an element in $K$ to be a SUBA to $f$ in $X$. This problem is analogous to the problem of determining those sets for which the so-called Kolmogorov

* This research was supported by NSERC Grant A8065 of Canada.



0021-9045/84 $3.00

criterion is a necessary condition for an element to be a best approximation to a given $f$. (See, e.g., [2].) Motivated by this analogy we give the

DEFINITION 1.1. A set $K$ is called a *strongly Kolmogorov set* if whenever $\pi$ is a SUBA to $f$ from $K$ then $\pi$ and $f$ must satisfy condition (1.1).

With the problem thus formulated, we can paraphrase the results of Wulbert, Bartelt and McLaughlin, and Papini by saying the class of strongly Kolmogorov sets includes the class of convex sets (and linear subspaces).

The following result, whose proof we will omit, shows that strongly Kolmogorov sets have a property that is analogous to being a sun.

PROPOSITION 1.2. *Let $K$ be strongly Kolmogorov and $\pi$ in $K$ be a* SUBA *to $f$. Then for every $0 \leqslant \lambda$, $\pi$ is also a* SUBA *to* $\pi + \lambda(f - \pi)$.

In Section 2 we give some characterizations of strongly Kolmogorov sets. However, our results are unsatisfactory in that all the characterizations are extrinsic. We also consider the question whether the class of strongly Kolmogorov sets is strictly larger than the class of convex sets.

In Section 3 we give an application to best approximation by monotone polynomials and show that best approximation by monotone polynomials in the $L_1$-sense is, in general, not strongly unique.

## 2. STRONGLY KOLMOGOROV SETS

In preparation of the theorems to be given, we give the following notations. We will denote the open ball centred at $z$ with radius $r$ by $B(z, r)$. For an arbitrary set $A$, con$(A)$ will be the cone (which need not be convex) generated by $A$. For any $z$, we define the supporting cone of the closed ball $B(0, \|z\|)$ at $z$ to be the set

$$K_z = \{ g: \operatorname{Re} \phi(g) \leqslant \|z\| \text{ for all } \phi \in \mathscr{S}_z \}.$$

Finally, recall the tangent functional $\tau(x, y)$ is defined by

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

The following result is known, and is easy to prove.

PROPOSITION 2.1. $\tau(x, y) = \sup_{\phi \in \mathscr{S}_x} \operatorname{Re} \phi(y).$

THEOREM 2.2. *Let $K$ be an arbitrary subset of a normed linear space $X$.*

*If $\pi$ in $K$ and $f$ in $X$ satisfy any one of the following set of conditions then they satisfy all of the conditions*:

(1) *There exists an $r > 0$ such that for all $g$ in $K$,*

$$\sup_{\phi \in \mathscr{L}_{f-\pi}} \operatorname{Re} \phi(\pi - g) \geqslant r \, \|\pi - g\|.$$

(2) *There exists an $r > 0$ such that for all $g$ in $K$,*

$$\tau(f - \pi, \pi - g) \geqslant r \, \|\pi - g\|.$$

(3) *$K_{f-\pi} \cap \operatorname{con}(\pi - K)$ is bounded.*

*Proof.* That (1) and (2) are equivalent follows from Proposition 2.1. We now prove (1) and (3) are equivalent. Suppose (1) holds. We shall prove that $K_{f-\pi} \cap \operatorname{con}(\pi - K)$ is contained in the ball $B(0, \|f - \pi\|/r)$. In fact, if there exists an $\alpha > 0$ and a $g \in K \sim \{\pi\}$ such that $\|\alpha(\pi - g)\| > \|f - \pi\|/r$ then

$$\begin{aligned} \sup_{\phi \in \mathscr{L}_{f-\pi}} \operatorname{Re} \phi(\alpha(\pi - g)) &= \alpha \sup_{\phi \in \mathscr{L}_{f-\pi}} \operatorname{Re} \phi(\pi - g) \\ &\geqslant \alpha r \, \|\pi - g\| \\ &> \|f - \pi\|. \end{aligned}$$

Thus $\alpha(\pi - g) \notin K_{f-\pi}$, so that

$$K_{f-\pi} \cap \operatorname{con}(\pi - K) \subset B(0, \|f - \pi\|/r).$$

Conversely, assume (3) holds. Then there is an $\alpha > 0$ for which $K_{f-\pi} \cap \operatorname{con}(\pi - K)$ is contained in the open ball $B(0, \alpha)$. Let $g \in K \sim \{\pi\}$ be arbitrary. Then

$$\alpha(\pi - g)/\|\pi - g\| \notin K_{f-\pi} \cap \operatorname{con}(\pi - K);$$

in fact, we have $\alpha(\pi - g)/\|\pi - g\| \notin K_{f-\pi}$. So there must be some $\phi_0$ in $\mathscr{L}_{f-\pi}$ for which

$$\operatorname{Re} \phi_0(\alpha(\pi - g)/\|\pi - g\|) > \|f - \pi\|,$$

that is,

$$\operatorname{Re} \phi_0(\pi - g) > \|f - \pi\| \, \|\pi - g\|/\alpha.$$

Let $r = \|f - \pi\|/\alpha$. Then $\sup_{\phi \in \mathscr{L}_{f-\pi}} \operatorname{Re} \phi(\pi - g) \geqslant r \, \|\pi - g\|$.

THEOREM 2.3. *The following set of conditions on a set $K$ are equivalent*:

(1) *K is a strongly Kolmogorov set.*

(2) *Whenever* $\pi \in K$ *is a* SUBA *to an* $f$ *there exists a constant* $r > 0$ *such that for all* $g$ *in* $K$,

$$\tau(f-\pi, \pi-g) \geqslant r\,\|\pi-g\|.$$

(3) *Whenever* $\pi$ *in* $K$ *is a* SUBA *to an* $f$, *the cone*

$$K_{f-\pi} \cap \operatorname{con}(\pi-K)$$

*is bounded.*

(4) *Whenever* $\pi$ *in* $K$ *is an* SUBA *to* $f$ *from* $K$ *there is then a uniform constant* $r = r(f) > 0$, *depending only on* $f$, *such that for every* $k$ *in* $K$ *and every* $g$ *in the convex hull of* $\pi$ *and* $k$ *we have*

$$\|f-g\| \geqslant \|f-\pi\| + r\,\|\pi-g\|.$$

*Proof.* The equivalence of (1), (2), and (3) follows from Theorem 2.2. So it remains to prove (1) and (4) are equivalent. Assume then (1) holds. Let $\pi$ be a SUBA to $f$ from $K$, and $k$ in $K$ be arbitrary. Since $K$ is a strongly Kolmogorov set there is a constant $r$ such that

$$\sup_{\phi \in \mathcal{J}_{f-\pi}} \operatorname{Re} \phi(\pi-k) \geqslant r\,\|\pi-k\|.$$

Now let $g = \lambda k + (1-\lambda)\pi$, $0 \leqslant \lambda \leqslant 1$. Then $\pi - g = \lambda(\pi-k)$ and so

$$\begin{aligned} \sup_{\phi \in \mathcal{J}_{f-\pi}} \operatorname{Re} \phi(\pi-g) &= \lambda \sup_{\phi \in \mathcal{J}_{f-\pi}} \operatorname{Re} \phi(\pi-k) \\ &\geqslant \lambda r\,\|\pi-k\| \\ &= r\,\|\pi-g\|. \end{aligned}$$

Thus, $\pi$ is a SUBA to $f$ and (4) follows.

Conversely, suppose (4) holds. Let $\pi$ be a SUBA to $f$ from $K$, and let $k$ in $K$ be arbitrary. Then by (4) there is a constant $r$, independent of $k$, such that

$$\|f-g\| \geqslant \|f-\pi\| + r\,\|\pi-g\|$$

for every $g$ in the convex hull of $\pi$ and $k$. Since convex sets are strongly Kolmogorov sets we have

$$\sup_{\phi \in \mathcal{J}_{f-\pi}} \operatorname{Re} \phi(\pi-g) \geqslant r\,\|\pi-g\|$$

for every $g$ in the convex hull of $\pi$ and $g$. By taking $g = k$ we see that (1) follows.

COROLLARY 2.4. *Let $K$ be a convex subset of a normed linear space. Then each of the conditions given in Theorem 2.2 are both necessary and sufficient for an element $\pi$ in $K$ to be a* SUBA *to $f$.*

*Remarks.* (i) It is clear that when $K$ is subspace then $\text{con}(\pi - k) = K$ for $\pi \in K$ so that condition (3) of Theorem 2.2 generalizes Bartelt and McLaughlin's result [1, p. 259] to arbitrary sets. In [5, p. 115] another generalization was given; but it is easy to see that the theorem is incorrect since it implies that best approximation from any compact set is always strongly unique.

(ii) Condition (1) of Theorem 2.2, when $K$ is a subspace, was given by Wulbert [7] for real fields and by Bartelt and McLaughlin for complex fields [1].

(iii) Conditions (2) and (1) of Theorem 2.2, when $K$ is convex, were given by Papini [5].

(iv) The conditions in Theorem 2.3 characterizing strongly Kolmogorov sets are all extrinsic and so it would be desirable to find some intrinsic characterizations, i.e., conditions that do not refer to the approximation problem.

A natural question that arises is whether the class of strongly Kolmogorov sets is strictly larger than the class of convex sets. The theorem and example that follow give some results related to this question.

DEFINITION 2.5. A set $K$ is strongly Chebyshev if for each $f$ there is a SUBA to $f$ from $K$.

THEOREM 2.6. *In a smooth normed linear space a set which is strongly Chebyshev and strongly Kolmogorov must be convex.*

*Proof.* Let $K$ be strongly Chebyshev and strongly Kolmogorov. Let $g_1$ and $g_2$ be in $K$ and suppose there is some $0 \leqslant \lambda \leqslant 1$ for which $f = \lambda g_1 + (1 - \lambda) g_2$ is not in $K$. Since $K$ is strongly Chebyshev there is an element $\pi$ in $K$ which is a SUBA to $f$. Since $K$ is strongly Kolmogorov, there are two hyperplanes passing through $\pi$ and separating $f$ and $g_i$, $i = 1, 2$. By smoothness, these two hyperplanes must coincide. But this leads to a contradiction because $f$, being on the line segment connecting $g_1$ and $g_2$, cannot be in the half-space opposite to the one that contains $g_1$ and $g_2$.

Wulbert proved that in a smooth normed linear space a best approximation from a proper linear subspace (which is not a singleton) cannot be strongly unique [7, p. 354]; consequently, there are no strongly Chebyshev subspaces in a smooth normed linear space. The following simple example shows that even in a Hilbert space a convex set can have a strongly unique element of best approximation.

EXAMPLE. Let $R^2$ be the plane with the $l_2$-norm. Let $K$ be the convex set

$$K = \{(x, y): |x| + |y| \leqslant 1\}.$$

Then using condition (3) of Theorem 2.2 it is easy to verify that $(1, 0)$ is a strongly unique best approximation to $(2, 0)$. This same example, however, shows there are elements of best approximation which are not strongly unique; for example, $(\frac{1}{2}, \frac{1}{2})$ is the best approximation to $(1, 1)$ which is not strongly unique. This fact can be verified easily using any of the conditions of Theorem 2.2 or using Theorem 2.7. For convenience we will assume the origin is an interior point of the convex set.

THEOREM 2.7. *Let $K$ be a closed convex subset of a Hilbert space and contain the origin as an interior point. If $\pi$ in $K$ is a point of smoothness of $K$ then $\pi$ is a best approximation which is not strongly unique.*

*Outline of Proof.* Since $\pi$ is a point of smoothness of $K$ there is a unique hyperplane $H$ supporting $K$ at $\pi$. Let $g$ be any vector orthogonal to $H$ and $f = g + \pi$. Then it is easy to verify that $\pi$ is the unique best approximation to $f$ from $K$. Since we are in a Hilbert space the set $\mathscr{L}_{f-\pi}$ is a singleton and so $K_{f-\pi}$ is a half-space. On the other hand, since $\pi$ is a point of smoothness of $K$, the origin 0 is a point of smoothness of $\pi - K$ so that $\text{con}(\pi - K)$ is a half-space. Thus $\text{con}(\pi - K) \cap K_{f-\pi}$ is unbounded; so $\pi$ cannot be strongly unique.

The following example shows that the class of strongly Kolmogorov sets is generally larger than the class of convex sets.

EXAMPLE. Let $R^2$ be the plane with the $\ell_\infty$-norm, i.e.,

$$\|(x, y)\| = \max\{|x|, |y|\}.$$

Let $K = \{(x, y): |x|^{1/2} + |y|^{1/2} \leqslant 1\}$. We let the reader verify that every point on the boundary of $K$ is an element of best approximation, and these are the only elements of best approximation. To show that $K$ is strongly Kolmogorov it is sufficient to verify that whenever $\pi$ in $K$ is a best approximation to $f$, the pair $\pi$ and $f$ satisfy the geometric condition (3) of Theorem 2.2. We shall do this only for a couple of special cases to give the reader a flavour of the verification process involved. For the first case we take $\pi = (\frac{1}{4}, \frac{1}{4})$ and $f = (\frac{5}{4}, \frac{5}{4})$. The geometric condition (3) of Theorem 2.2 is given by Fig. 1. In this case

$$K_{f-\pi} = \{(x, y): x \leqslant 1 \text{ and } y \leqslant 1\}$$

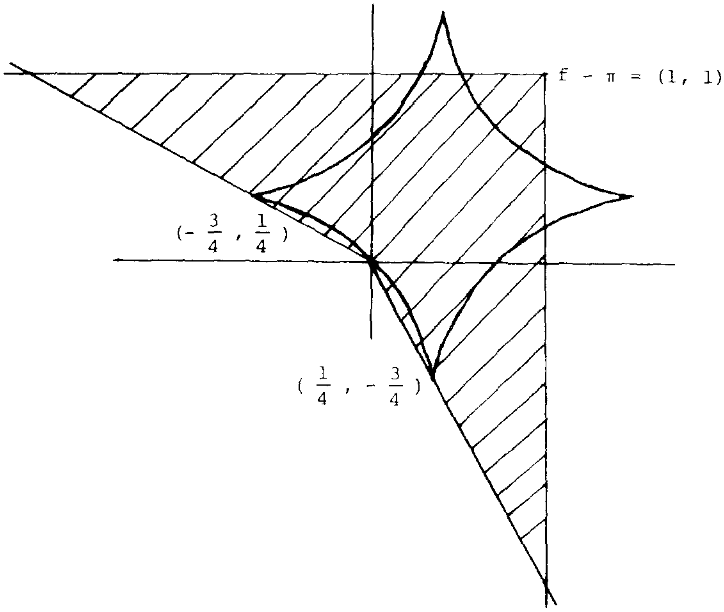


FIG. 1. $K_{f-\pi} \cap \mathrm{con}(\pi - K)$.

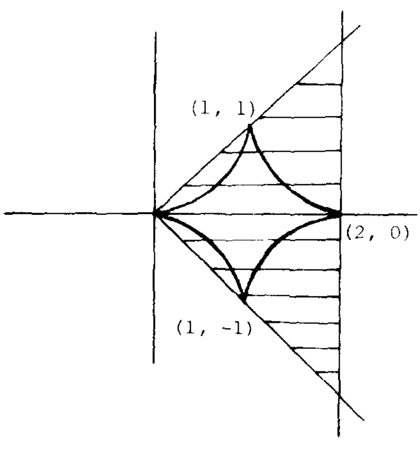


FIG. 2. $K_{f-\pi} \cap \mathrm{con}(\pi - K)$.

and $\text{con}(\pi - K)$ is the region generated by rotating the ray through $(\frac{1}{4}, -\frac{3}{4})$ counterclockwise to the ray through $(-\frac{3}{4}, \frac{1}{4})$. The set $K_{f-\pi} \cap \text{con}(\pi - K)$ is the shaded region in Fig. 1. For the second case, we take $\pi = (1, 0)$ and $f = (3, 0)$. The set $K_{f-\pi}$ corresponding to this case is given by

$$K_{f-\pi} = \{(x, y): x \leqslant 2\}$$

and $\text{con}(\pi - K)$ is the region generated by rotating the ray through $(1, -1)$ counterclockwise to the ray through $(1, 1)$. The set $K_{f-\pi} \cap \text{con}(\pi - K)$ is given by the shaded region in Fig. 2.

## 3. An Application to Approximation by Monotone Polynomial

Throughout this section $C[a, b]$ will denote the set of real-valued continuous functions defined on the interval $[a, b]$, and $\mathscr{P}_n$ will denote the set of all polynomials whose degree is at most $n$. Let $1 \leqslant k_1 < \cdots < k_m \leqslant n$ be fixed integers and let $\varepsilon_i = \pm 1$, $i = 1,..., m$. Define the set of "monotone" polynomials to be the convex cone

$$K = \{ p \in \mathscr{P}_n : \varepsilon_i p^{(k_i)}(x) \geqslant 0, a \leqslant x \leqslant b, i = 1,..., m\}.$$

(Here $p^{(i)}$ denotes the $i$th derivative of $p$.) In a surprise result Fletcher and Roulier [3] showed recently that with the uniform norm, best approximation by monotone polynomials need not be strongly unique. It is our intention to show that when $C[a, b]$ is equipped with the $L_1$-norm, i.e.,

$$\|f\| = \int_a^b |f(t)| \, dt,$$

best approximation by monotone polynomials need not be strongly unique either.

Theorem 3.1. *There is a function $f$ in $C[a, b]$ for which its best $L_1$-approximation from $K$ is not strongly unique.*

*Proof.* Let $g$ be a polynomial whose degree exceeds $n$. Then by the classical theorem of Jackson there is a unique polynomial $p$ in $\mathscr{P}_n$ which best approximates $g$. Let $f = g - p$. Then 0 is the best approximation to $f$ from $\mathscr{P}_n$. Since $K$ is a subset of $\mathscr{P}_n$, 0 must be the best approximation to $f$ from $K$ also.

Now $f$ is a polynomial and so $f \neq 0$ a.e. Thus $f$ is a point of smoothness of the closed ball $B(0, \|f\|)$ [4, p. 350]. Consequently, $\mathscr{L}_f$ consists of a singleton, say $\phi$. Thus $K_f$ is a half-space. By [6, p. 18] $\phi$ must annihilate $\mathscr{P}_n$, so that $\mathscr{P}_n$ is contained in $K_f$. Consequently,

$$K_f \cap \operatorname{con}(-K) = K_f \cap (-K) = -K.$$

This shows $K_f \cap \operatorname{con}(-K)$ is unbounded so that 0 cannot be strongly unique.

## ACKNOWLEDGMENTS

The author wishes to record his thanks to Dr. Ed Leonard and Dr. John Whitfield for several helpful conversations.

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